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Research Article

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Ramsey numbers of partial order graphs (comparability graphs) and implications in ring theory

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Abstract: For a partially ordered set $(A, \leq)$, let G_A be the simple, undirected graph with vertex set A such that two vertices $a \neq b \in A$ are adjacent if either $a \leq b$ or $b \leq a$. We call G_A the *partial order graph* or *comparability graph* of A. Furthermore, we say that a graph G is a partial order graph if there exists a partially ordered set A such that $G = G_A$. For a class C of simple, undirected graphs and $n, m \geq 1$, we define the *Ramsey number* $\mathcal{R}_C(n, m)$ with respect to C to be the minimal number of vertices r such that every induced subgraph of an arbitrary graph in C consisting of r vertices contains either a complete n-clique K_n or an independent set consisting of m vertices. In this paper, we determine the Ramsey number with respect to some classes of partial order graphs. Furthermore, some implications of Ramsey numbers in ring theory are discussed.

Keywords: Ramsey number, partial order, partial order graph, inclusion graph

MSC 2020: 13A15, 06A06, 05CXX, 05D10

1 Introduction

The Ramsey number $\mathcal{R}(n, m)$ gives the solution to the party problem, which asks for the minimum number $\mathcal{R}(n, m)$ of guests that must be invited so that at least n will know each other or at least m will not know each other. In the language of graph theory, the Ramsey number is the minimum number $v = \mathcal{R}(n, m)$ of vertices such that all undirected simple graphs of order v contain a clique of order n or an independent set of order m. There exists a considerable amount of literature on Ramsey numbers. For example, Greenwood and Gleason [1] showed that $\mathcal{R}(3, 3) = 6$, $\mathcal{R}(3, 4) = 9$ and $\mathcal{R}(3, 5) = 14$; Graver and Yackel [2] proved that $\mathcal{R}(3, 6) = 18$; Kalbfleisch [3] computed that $\mathcal{R}(3, 7) = 23$; McKay and Min [4] showed that $\mathcal{R}(3, 8) = 28$ and Grinstead and Roberts [5] determined that $\mathcal{R}(3, 9) = 36$.

A summary of known results up to 1983 for $\mathcal{R}(n, m)$ is given in the study by Chung and Grinstead [6]. An up-to-date-list of the best currently known bounds for generalized Ramsey numbers (multicolor graph numbers), hypergraph Ramsey numbers and many other types of Ramsey numbers is maintained by Radziszowski [7].

In this paper, we determine the Ramsey number of partial order graphs. We want to point out that recently, a colleague kindly made us aware that such graphs in the literature are also known as comparability graph and our result Theorem 2.2 is a consequence of [8, Theorem 6] (also see [8, Corollary 1]).

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However, our proof of Theorem 2.2 is self-contained and it is completely different from the proof in [8]. Our proof solely relies on the pigeonhole principle. For a partially ordered set $(A, \leq)$, let G_A be the simple, undirected graph with vertex set A such that two vertices $a \neq b \in A$ are adjacent if either $a \leq b$ or $b \leq a$. We call G_A the *partial order graph* (*comparability graph*) of A. In this paper, we will just use the name partial order graph. Furthermore, we say that a graph G is a partial order graph if there exists a partially ordered set A such that $G = G_A$. For a class C of simple, undirected graphs and $n, m \geq 1$, we define the *Ramsey number* $\mathcal{R}_C(n, m)$ with respect to the class C to be the minimal number r of vertices such that every induced subgraph of an arbitrary graph in C consisting of r vertices contains either a complete n-clique K_n or an independent set consisting of m vertices.

Next, we remind the readers of the graph theoretic definitions that are used in this paper. We say that a graph G is *connected* if there is a path between any two distinct vertices of G. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\operatorname{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G ($g(G) = \infty$ if G contains no cycles). We denote the *complete graph* on n vertices or *n-clique* by K_n and the *complete bipartite* graph on m and n vertices by $K_{m,n}$. The *clique number* $\omega(G)$ of G is the largest positive integer m such that K_m is an induced subgraph of G. The *chromatic number* of G, $\chi(G)$, is the minimum number of colors needed to produce a proper coloring of G (that is, no two vertices that share an edge have the same color). The *domination number* of G, $\gamma(G)$, is the minimum size of a set S of vertices of G such that each vertex in $G \setminus S$ is connected by an edge to at least one vertex in S by an edge. An *independent vertex* set of G is a subset of the vertices such that no two vertices in the subset are connected by an edge of G. For a general reference for graph theory we refer to Bollobás' textbook [9].

In Section 2, we show that the Ramsey number $\mathcal{R}_{\mathcal{PoG}}(n, m)$ for the class $\mathcal{PoG}$ of partial order graphs equals $(n-1)(m-1)+1$, see Theorem 2.2. In Section 3, we study subclasses of partial order graphs that appear in the context of ring theory. Among other results, we show that for the classes $\mathcal{PDG}$ of perfect divisor graphs, $\mathcal{D}iv\mathcal{G}$ of divisibility graphs, $In\mathcal{G}$ of inclusion ideal graphs, $\mathcal{M}at\mathcal{G}$ of matrix graphs and $Idem\mathcal{G}$ of idempotent graphs of rings, the respective Ramsey numbers equal to $\mathcal{R}_{\mathcal{PoG}}$, see Theorems 3.4, 3.8, 3.12, 3.16 and 3.21, respectively. In Section 4, we a present a subclass of partial ordered graphs with respect to which the Ramsey numbers are non-symmetric.

Throughout this paper, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integer modulo n, respectively. Moreover, for a ring R we assume that $1 \neq 0$ holds, $R^\bullet = R \setminus \{0\}$ denotes the set of non-zero elements of R and $U(R)$ denotes the group of units of R.

2 Ramsey numbers of partial order graphs

Definition 2.1.

(1) For a partially ordered set $(A, \leq)$, let G_A be the simple, undirected graph with vertex set A such that two vertices $a \neq b \in A$ are adjacent if either $a \leq b$ or $b \leq a$. We call G_A the *partial order graph* of A. Furthermore, we say that G is a partial order graph if there exists a partially ordered set A such that $G = G_A$. By $\mathcal{PoG}$ we denote the *class of all partial order graphs*.

(2) For a class C of simple, undirected graphs and $n, m \geq 1$, we set $\mathcal{R}_C(n, m)$ to be the minimal number r of vertices such that every induced subgraph of an arbitrary graph in C consisting of r vertices contains either a complete n-clique K_n or an independent set consisting of m vertices. We call $\mathcal{R}_C$ the *Ramsey number with respect to the class C*.

Theorem 2.2. *Let $n, m \geq 1$ (n, m need not be distinct). Then for the Ramsey number $\mathcal{R}_{\mathcal{PoG}}$ with respect to the class $\mathcal{PoG}$ of partial order graphs, the following equality holds*

$$\mathcal{R}_{\mathcal{PoG}}(n, m) = \mathcal{R}_{\mathcal{PoG}}(m, n) = (n-1)(m-1)+1.$$

Proof. First, we prove that $\mathcal{R}_{\mathcal{P}o\mathcal{G}}(n, m) > (n-1)(m-1)$. Let A be a set of cardinality $(n-1)(m-1)$ and $A_1, \ldots, A_{n-1}$ an arbitrary partition of A into $n-1$ subsets each of cardinality $m-1$. Furthermore, for $a, b \in A$, we say $a \preccurlyeq b$ if and only if $a = b$ or $a \in A_i$ and $b \in A_j$ with $i < j$. Then $\preccurlyeq$ is a partial order on A and the partial order graph G_A is a complete $(n-1)$-partite graph in which each partition has $m-1$ independent vertices. It is easily verified that the clique number of G_A is $n-1$ and that at most $m-1$ vertices of G_A are independent.

Let G be a partial order graph and H an induced subgraph. We show that if H contains $(n-1)(m-1)+1$ vertices, then H contains either an n-clique K_n or an independent set of m vertices.

Let G^{dir} be the directed graph with the same vertex set as G such that (a, b) is an edge if $a \neq b$ and $a \leq b$. Then H^{dir} (the subgraph of G^{dir} induced by the vertices of H) contains a directed path of length n if and only if H contains an $(n+1)$-clique K_{n+1}.

Note that G^{dir} does not contain a directed cycle. This allows us to define $\text{pos}_H(a)$ to be the maximal length of a directed path in H^{dir} with endpoint a for a vertex a of H.

It is easily seen that $\text{pos}_H(b) \leq \text{pos}_H(a) - 1$ for every edge (b, a) in H^{dir}. In particular, if for two vertices a, b of H, $\text{pos}_H(a) = \text{pos}_H(b)$, then the two vertices are independent in H.

Moreover, a straight-forward argument shows that H contains an n-clique K_n if and only if there exists a vertex a in H with $\text{pos}_H(a) \geq n-1$.

Now, assume that H does not contain an n-clique K_n. This implies that $\text{pos}_H(a) < n-1$ for all vertices a in H. It then follows by the pigeonhole principle that among the $(n-1)(m-1)+1$ vertices in H, there are at least m vertices a with $\text{pos}_H(a) = k$ for some k, $0 \leq k \leq n-2$. Therefore, H contains m independent vertices.

Since $(n-1)(m-1)+1$ is symmetric in n and m, it further follows that $\mathcal{R}_{Po\mathcal{G}}(n, m) = \mathcal{R}_{Po\mathcal{G}}(m, n)$. □

3 Subclasses of partial order graphs that appear in the ring theory

In this section, we discuss subclasses of partial order graphs that appear in the context of ring theory. In particular, we focus on the implications of Theorem 2.2. Recall for a class C of graphs, $\mathcal{R}_C$ denotes the Ramsey number with respect to C, cf. Definition 2.1.

3.1 Perfect divisor graphs

Definition 3.1. Let R be a commutative ring, $n \in \mathbb{N}_{\geq 2}$ and $S = \{m_1, \ldots, m_n\} \subseteq R^\bullet \backslash U(R)$ be a set of n pairwise coprime non-zero non-units and $m = m_1 m_2 \cdots m_n$. (Note that $m = 0$ is possible.)

(1) We say d is a *perfect divisor* of m with respect to S if $d \neq m$ and d is a product of distinct elements of S.
(2) The *perfect divisor graph PDG*(S) of S is defined as the simple, undirected graph (V, E), where $V = \{d \mid d \text{ perfect-divisor of } m\}$ is the vertex set and for two vertices $a \neq b \in V$, $(a, b) \in E$ if and only if $a|b$ or $b|a$.
(3) By $\mathcal{PDG}$ we denote the *class of all perfect divisor graphs*.

Lemma 3.2. *Let R be a commutative ring, $n \in \mathbb{N}_{\geq 2}$ and $S = \{m_1, \ldots, m_n\} \subseteq R^\bullet \backslash U(R)$ be a set of n pairwise coprime non-zero non-units and $m = m_1 m_2 \cdots m_n$. Furthermore, let*

$$V = \{d \mid d \text{ perfect divisor of } m \text{ with respect to } S\}$$

and define $\leq$ on V such that for all $a, b \in V$, we have $a \leq b$ if and only if $a = b$ or $a|b$.

Then $(V, \leq)$ is a partially ordered set of cardinality $|V| = 2^n - 2$ and PDG(S) *is a partial order graph.*

Proof. The relation $\leq$ clearly is reflexive and transitive, we prove that it is also antisymmetric. Let $d \in V$ be a perfect divisor of m with respect to S. Then $d = \prod_{j \in J} m_j$ for $\varnothing \neq J \subseteq \{1, \ldots, n\}$. We show that for every $1 \leq i \leq m$, $m_i | d$ if and only if $i \in J$.

Obviously, if $j \in J$, then $m_i | d$. Let us assume that $i \in \{1,\ldots,n\} \setminus J$. Then by hypothesis, for $j \in J$ there are elements a_j and $b_j \in R$ such that $a_j m_j + b_j m_i = 1$ holds. Hence,

$$1 = \prod_{j \in J} (a_j m_j + b_j m_i) = \left(\prod_{j \in J} a_j m_j \right) + c m_i = ad + c m_i$$

for some $a, c \in R$. Therefore, d and m_i are coprime elements of R which in particular implies that $m_i \nmid d$.

It follows that if d_1 and d_2 are distinct perfect divisors of m and $d_1 | d_2$, then $d_2 \nmid d_1$. Thus, $(V, \leq)$ is a partially ordered set.

Moreover, it follows that the elements in V correspond to the non-empty proper subset of $\{1,\ldots,n\}$. Therefore, their number amounts to

$$|V| = |\{\emptyset \neq J \subsetneq \{1,\ldots,n\}\}| = \sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2. \qquad \square$$

Theorem 3.3. *Let R be a commutative ring, $n \in \mathbb{N}_{\geq 2}$ and $S = \{m_1,\ldots,m_n\} \subseteq R^{\bullet} \setminus U(R)$ be a set of n pairwise coprime non-zero non-units, $m = m_1 m_2 \cdots m_n$ and* PDG(S) *the perfect divisor graph of m with respect to S.*

Then the following assertions hold:

(1) PDG(S) *is a connected graph if and only if $n \geq 3$.*
(2) *If $n \geq 3$, then the diameter* diam(PDG(S)) $= 3$.
(3) *The domination number of* PDG(S) *is equal to* 2 *if $n \geq 2$ and equal to* 1 *if $n = 1$.*
(4) *If $n \geq 3$, then the vertices in $P_k = \{\prod_{j \in J} m_j \mid |J| = k\}$ for $1 \leq k \leq n-1$ are pairwise not connected by an edge. In particular,* PDG(S) *is an $(n-1)$-partite graph.*
(5) *If $a \in P_k = \{\prod_{j \in J} m_j \mid |J| = k\}$ for $1 \leq k \leq n-1$, then* $\deg(a) = 2^k + 2^{n-k} - 4$.
(6) *If $n \geq 3$, then for the girth of* PDG(S) *the following holds*

$$g(\mathrm{PDG}(S)) = \begin{cases} 6 & n = 3, \\ 3 & n \geq 4. \end{cases}$$

(7) PDG(S) *is planar if and only if $n \in \{3, 4\}$.*

Proof.

(1) If $n = 2$, then V consists of two vertices m_1 and m_2 which are coprime and hence not connected. Assume $n \geq 3$ and let $a = \prod_{j \in J} m_j$ and $b = \prod_{k \in K} m_k$ be two distinct vertices of PDG(S). Suppose that $m_j = m_k$ for some $j \in J$ and $k \in K$. Then $a - m_j - b$ is a path of length 2 from a to b if $m_j \neq a, b$ and (a, b) is an edge otherwise. Suppose that $m_j \neq m_k$ for every $j \in J$ and $k \in K$. We show that $|J| \leq n-2$ or $|K| \leq n-2$. Suppose that $|J| = |K| = n-1$. Since $n \geq 3$ and $m_j \neq m_k$ for every $j \in J$ and $k \in K$, we conclude that $|\{m_j \mid j \in J\} \cup \{m_k \mid k \in K\}| = 2n - 2 > n$, a contradiction. Thus, $|J| \leq n-2$ or $|K| \leq n-2$. Without loss of generality, we may assume that $|J| \leq n-2$. Take arbitrary $k \in K$. Then, $a - am_k - m_k - b$ is a path of length 3 from a to b if $b \neq m_k$ and otherwise, $a - am_k - b$ is a path of length 2. Hence, PDG(S) is connected which completes the proof of (1).

(2) Suppose that $n \geq 3$. Then PDG(S) is connected by (1). Let a, b be two distinct vertices of PDG(S). In light of the proof given in (1), we have $d(a, b) \leq 3$. Let $a = \prod_{j=1}^{(n-1)} m_j$ and $b = m_n$. Then $a - m_1 - m_1 b - b$ is a shortest path in PDG(S) from a to b. Hence, $d(a, b) = 3$. Thus, diam(PDG(S)) $= 3$.

For (3) observe that every perfect divisor d of m is either divisible by m_1 or divides $m_2 m_3 \cdots m_n$. Hence, every vertex of PDG(S) is connected by an edge to either one of these two vertices.

(4) Let $1 \leq k \leq n-1$ and $J, K \subseteq \{1,\ldots,n\}$ with $|J| = |K| = k$. Set $a = \prod_{j \in J} m_j$ and $b = \prod_{k \in K} m_k$ be two different vertices of PDG(S), which implies $J \neq K$. Therefore, there exist $j \in J \setminus K$ and $k \in K \setminus J$. In the proof of Lemma 3.2, we have shown that it now follows that $m_j \nmid b$ and $m_k \nmid a$. In particular, it follows that $a \nmid b$ and $b \nmid a$. Hence, no two vertices in $\{\prod_{j \in J} m_j \mid |J| = k\}$ are connected by an edge.

For (5), let $a = \prod_{j \in J} m_j$ be perfect divisor of m and set $k = |J|$. The perfect divisors of m with respect to S which divide a which are connected by an edge to a correspond to the non-empty, proper subsets of J which are $\sum_{i=1}^{k-1} \binom{k}{i} = 2^k - 2$ many. In addition, we need to count the number of perfect divisors of m which are divisible by a. These are exactly the ones of the form $\prod_{k\in K} m_k$ with $J \subsetneq K \subsetneq \{1,\dots,n\}$ of which there are $\sum_{i=1}^{n-k-1}\binom{n-k}{i} = 2^{n-k} - 2$. Hence, $\deg(a) = 2^k + 2^{n-k} - 4$.

(6) For $n = 3$, we can verify in Figure 1 that there is cycle of length 6 and no shorter cycle.

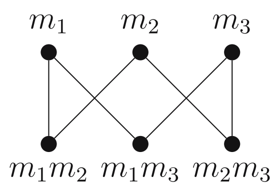

Figure 1: Perfect divisor graph for $n = 3$.

If $n \geq 4$, then $m_1m_2m_3$ is a perfect divisor and the edges (m_1, m_1m_2), $(m_1m_2, m_1m_2m_3)$ and $(m_1m_2m_3, m_1)$ form a cycle of length 3 which is the smallest possible length of a cycle in PDG(S).

Finally, for (7), it is easily verified that PDG(S) is planar if $n = 3$, cf. Figure 1. Moreover, Figure 2 shows a planar arrangement of the edges of PDG(S) for $n = 4$.

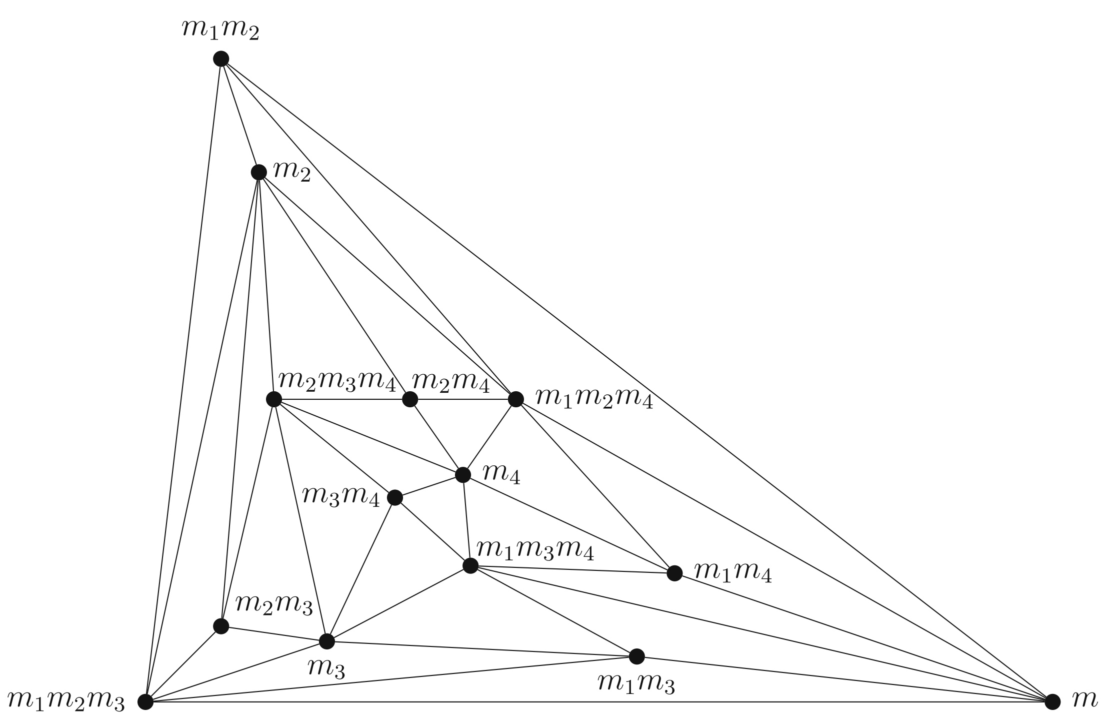

Figure 2: Perfect divisor graph for $n = 4$ with planar arrangement of edges.

If, however, $n \geq 5$, then Figure 3 is a $K_{3,3}$ subgraph of PDG(S), and hence PDG(S) is not a planar by Kuratowski's Theorem on planar graphs. □

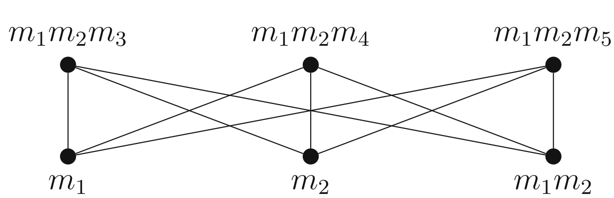

Figure 3: PDG(S) contains $K_{3,3}$ as a subgraph for $n \geq 5$.

Next, we compute the Ramsey number with respect to the class of perfect divisor graphs. Note that $\mathcal{PDG}$ is a subclass of $\mathcal{PoG}$ which immediately implies that $\mathcal{R}_{\mathcal{PDG}}(n,m) \le \mathcal{R}_{\mathcal{PoG}}(n,m)$ for all $n, m \ge 1$. We use Theorem 3.3 to show that equality holds.

Theorem 3.4. *Let $n, m \ge 1$. Then for the Ramsey number $\mathcal{R}_{\mathcal{PDG}}$ with respect to the class $\mathcal{PDG}$ of perfect divisor graphs the following holds:*

$$\mathcal{R}_{\mathcal{PDG}}(n,m) = \mathcal{R}_{\mathcal{PoG}}(n,m) = (n-1)(m-1)+1.$$

Proof. We set $w = (n-1)(m-1)$ and show that $\mathcal{R}_{\mathcal{PDG}}(n,m) > w = (n-1)(m-1)$ by giving an example of perfect divisor graph G and an induced subgraph H of G with w vertices which is a complete $(n-1)$-partite graph on w vertices in which independent sets are of cardinality at most $m-1$.

Let $R = \mathbb{Z}$ and let $S = \{p_1, p_2, \ldots, p_w\}$ be a set of w distinct positive prime numbers of $\mathbb{Z}$. We set $m = p_1 p_2 \cdots p_w$ and $G = \mathrm{PDG}(S)$.

For each $1 \le i \le n-1$, let $k_i = (i-1)(m-1)$ and we set $a_i = p_1 p_2 \cdots p_{k_i}$ (where $a_1 = 1$) and

$$A_i = \{a_i p_{k_i+1}, \ldots, a_i p_{k_i+(m-1)}\}.$$

Note that $A_1 = \{p_1, \ldots, p_{m-1}\}$.

Let H be the subgraph of G induced by the vertex set $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$. By construction, for each $1 \le i \le n-1, |A_i| = m-1$ holds and A_i is contained in the partition P_{k_i+1} of G, cf. Theorem 3.3.4. This implies that each A_i is an independent vertex set of H of cardinality $m-1$.

Moreover, since G is a $(w-1)$-partite graph and each A_i is contained in P_{k_i+1}, it follows that H is an $(n-1)$-partite graph (with partitioning $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$). For an example of this construction with $m = 5$ and $n = 4$ see Example 3.5.

Thus, not more than $m-1$ vertices of H are independent, and a straight-forward verification shows that the clique number of H is at most $n-1$. Thus, $\mathcal{R}_{\mathcal{PDG}}(n,m) > w$. Hence, by Theorem 2.2, we have $\mathcal{R}_{\mathcal{PDG}}(n,m) = \mathcal{R}_{\mathcal{PoG}}(n,m) = w+1 = (n-1)(m-1)+1$. □

Example 3.5. We demonstrate the construction of the previous proof for the example $R = \mathbb{Z}$ with $n = 4$ and $m = 5$. That is, we construct a perfect divisor graph which has a complete 3-partite graph H as subgraph and each of the partitions of H consist of four independent vertices.

Let $w = (n-1)(m-1) = 12$ and we set $S = \{p_1, p_2, \ldots, p_{12}\}$. Next, let $n_i = (i-1)(m-1)$ for $1 \le i \le 3$, that is, $n_1 = 0$, $n_2 = 4$ and $n_3 = 8$. Then $a_1 = 1$, $a_2 = p_1p_2p_3p_4$ and $a_3 = p_1 p_2 \cdots p_8$.

We set

$$\begin{aligned}
A_1 &= \{a_1p_1, a_1p_2, a_1p_3, a_1p_4\} = \{p_1, p_2, p_3, p_4\},\\
A_2 &= \{a_2p_5, a_2p_6, a_2p_7, a_2p_8\} = \{(p_1\cdots p_4)p_5, (p_1\cdots p_4)p_6, (p_1\cdots p_4)p_7, (p_1\cdots p_4)p_8\},\\
A_3 &= \{a_3p_9, a_3p_{10}, a_3p_{11}, a_3p_{12}\} = \{(p_1p_2\cdots p_8)p_9, (p_1p_2\cdots p_8)p_{10}, \ldots, (p_1p_2\cdots p_8)p_{12}\}.
\end{aligned}$$

The subgraph of $\mathrm{PDG}(S)$ induced by $A_1 \cup A_2 \cup A_3$ is a complete 3-partite graph in which each partition has four vertices that are independent (Figure 4).

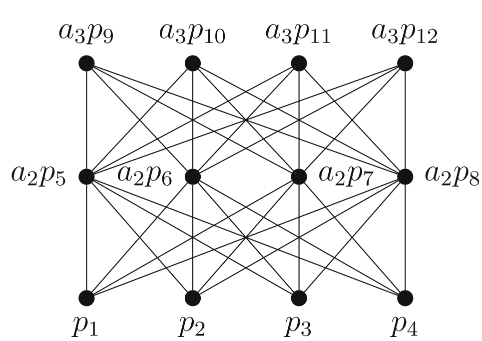

Figure 4: Induced subgraph H of $\mathrm{PDG}(\{p_1, \ldots, p_{12}\})$ where, for better visibility, the edges between A_1 and A_3 are “hidden” behind the edges between A_1 and A_2 and the edges between A_2 and A_3.

3.2 The divisibility graph of a commutative ring

Definition 3.6. Let R be a commutative ring and a, b be distinct elements of R.
(1) If a is a non-zero non-unit element of R, then we say a is a *proper* element of R.
(2) If $a|b$ (in R) and $b\nmid a$ (in R), then we write $a||b$.
(3) The *divisibility graph* $\mathrm{Div}(R)$ of R is the undirected simple graph whose vertex set consists of the proper elements of R such that two vertices $a \neq b$ are adjacent if and only if $a||b$ or $b||a$.

The following lemma can be verified by a straight-forward argument.

Lemma 3.7. *Let R be a commutative ring and let V be the set of all proper elements of R and define $\leq$ on V such that for all $a, b \in V$, we have $a \leq b$ if and only if $a = b$ or $a||b$.*
Then $(V, \leq)$ is a partially ordered set and the divisibility graph $\mathrm{Div}(R)$ of R is a partial order graph.

By Lemma 3.7, it is clear that $\mathcal{R}_{DivG}(n, m) \leq \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n, m)$ holds. However, since a perfect divisor graph is an induced subgroup of a divisibility graph, it follows from Theorem 3.4 that equality holds. We conclude the following theorem.

Theorem 3.8. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct). Then for the Ramsey number $\mathcal{R}_{\mathcal{D}iv\mathcal{G}}$ with respect to the class $\mathcal{D}iv\mathcal{G}$ of divisibility graphs the following holds:*

$$\mathcal{R}_{\mathcal{D}iv\mathcal{G}}(n, m) = \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n, m) = (n-1)(m-1) + 1.$$

Moreover, in view of Theorem 3.8, we have the following result.

Corollary 3.9. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct), $k = (n-1)(m-1) + 1$, R be a commutative ring and S be a subset of proper elements of R such that $|S| \geq k$.*
Then one of the following assertions holds:
(1) *There are n elements $a_1, \ldots, a_n \in S$ such that $a_1||a_2||\cdots||a_n$ (in R).*
(2) *There are m pairwise distinct elements $b_1, \ldots, b_m \in S$ such that for all $1 \leq h \neq f \leq m$ either*
- *$b_h \nmid b_f$ or*
- *$b_h|b_f$ and $b_f|b_h$*

hold.

3.3 Inclusion ideal graphs of rings

Definition 3.10. Let R be a ring.
(1) We call a left (right) ideal I of R *non-trivial* if $I \neq \{0\}$ and $I \neq R$.
(2) The *inclusion ideal graph* $\mathrm{In}(R)$ of R is the (simple, undirected) graph whose vertex set is the set of non-trivial left ideals of R and two distinct left ideals I, J are adjacent if and only if $I \subset J$ or $J \subset I$ (cf. Akbari et al. [10]).
(3) By $In\mathcal{G}$, we denote the *class of all inclusion ideal graphs.*

Remark 3.11. The set V of all non-trivial left ideals of a ring R together with the partial order $\subseteq$ induced by inclusion is a partially ordered set. Hence, the inclusion graph $\mathrm{In}(R)$ of a ring R is a partial order graph.

By Remark 3.11, it is clear that $\mathcal{R}_{In\mathcal{G}}(n, m) \leq \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n, m)$. The reverse inequality can be seen from the following argument. Let G be the graph constructed in the proof of Theorem 3.4, that is, $G = \mathrm{PDG}(S)$ with $S = \{p_1, \ldots, p_w\}$ is the set of $w = (n-1)(m-1)$ distinct positive primes of $\mathbb{Z}$. Recall that G contains a subgraph with $(n-1)(m-1)$ vertices whose clique number is at most $n-1$ and in which not more than $m-1$ vertices are independent. The graph G is graph-isomorphic to a subgraph of the inclusion ideal graph of $\mathbb{Z}$, namely, the subgraph induced by the principal ideals generated by the elements in the vertex set of G. Since the inclusion ideal graph of $\mathbb{Z}$ is contained in InG, it follows that $(n-1)(m-1) < \mathcal{R}_{In\mathcal{G}}(n,m)$. Hence, by Theorem 2.2 we conclude the following theorem.

Theorem 3.12. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct). Then for the Ramsey number $\mathcal{R}_{In\mathcal{G}}$ with respect to the class $In\mathcal{G}$ of inclusion ideal graphs the following holds:*

$$\mathcal{R}_{In\mathcal{G}}(n,m) = \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n,m) = (n-1)(m-1)+1.$$

In view of Theorem 3.12, we have the following result.

Corollary 3.13. *Let R be a ring, $n, m \geq 1$ be positive integers (n, m need not be distinct) and $S \subseteq \{I \mid I$ is a non-trivial left ideal of $R\}$ such that $|S| \geq (n-1)(m-1)+1$.*

Then one of the following assertions hold:

(1) *There are n pairwise distinct elements (non-trivial left ideals) $I_1, \ldots, I_n \in S$ with $I_1 \subset I_2 \subset \cdots \subset I_n$.*
(2) *There are m elements (non-trivial left ideals) $J_1, \ldots, J_m \in S$ such that $J_a \nsubseteq J_b$ for every $1 \leq a \neq b \leq m$.*

3.4 Matrix graphs over commutative rings

Definition 3.14. Let R be a commutative ring which is not a field and $j \geq 2$ an integer.

(1) We denote by $R^{j\times j}$ the ring of all $j \times j$ matrices with entries in R.
(2) Let $V = \{A \in R^{j\times j} \mid \det(A)$ a proper element of $R\}$ be the set of all $j \times j$ matrices whose determinant is a proper element of R, cf. Definition 3.6. We define the *matrix graph* $\mathrm{Mat}G(R)$ of R to be the undirected simple graph with V as its vertex set and two distinct vertices $A, B \in V$ are adjacent if and only if $\det(A) \,||\, \det(B)$ or $\det(A) \,||\, \det(B)$.
(3) By $\mathcal{M}at\mathcal{G}$ we denote the *class of all matrix graphs*.

Lemma 3.15. *Let R be a commutative ring which is not a field, $j \geq 2$ an integer and*

$$V = \{A \in R^{j\times j} \mid \det(A) \text{ is a proper element of } R\}.$$

Define $\leq$ on V such that for all $A, B \in V$, we have $A \leq B$ if and only if $A = B$ or $\det(A) \,||\, \det(B)$.

Then $(V, \leq)$ is a partially ordered set and the graph $\mathrm{Mat}G(R)$ is a partial order graph.

By Theorem 2.2, it is clear that $\mathcal{R}_{\mathcal{M}at\mathcal{G}}(n,m) \leq \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n,m)$. We prove next that equality holds.

Theorem 3.16. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct). Then for the Ramsey number $\mathcal{R}_{\mathcal{M}at\mathcal{G}}$ with respect to the class $\mathcal{M}at\mathcal{G}$ of matrix graphs the following holds:*

$$\mathcal{R}_{\mathcal{M}at\mathcal{G}}(n,m) = \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n,m) = (n-1)(m-1)+1.$$

Proof. Let $R = \mathbb{Z}$ and $j \geq 2$ and set $w = (n-1)(m-1) \geq 1$. Furthermore, let $p_1, p_2, \ldots, p_w$ be distinct positive prime numbers of $\mathbb{Z}$ and choose $X_i \in R^{j\times j}$ with $\det(X_i) = p_i$ for $1 \leq i \leq w$.

We construct a matrix graph $\mathrm{Mat}G(R)$ which has a complete $(n-1)$-partite subgraph H in which each partition has $m-1$ vertices. The construction is analogous to the one in the proof of Theorem 3.4.

For each $1 \leq i \leq n-1$, let $k_i = (i-1)(m-1)$, $q_i = X_1X_2\cdots X_{k_i}$ (hence $q_1 = I_j$ the identity matrix $j \times j$) and

$$A_i = \{q_iX_{k_i+1}, \ldots, q_iX_{k_i+(m-1)}\}.$$

Note that $A_1 = \{X_1, \ldots, X_{m-1}\}$. Since $\det(q_iX_{k_i+j}) = p_1 \ldots p_{k_i}p_{k_i+j}$, it follows that the elements of A_i are pairwise distinct and $|A_i| = m-1$ for $1 \leq i \leq n-1$.

Let $S = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ and set $G = \mathrm{Mat}G(\mathbb{Z})$. Then for each i, the vertices in A_i are independent. However, there are edges between all vertices of two distinct sets A_i and A_j with $i \neq j$. Therefore, G is a complete $(n-1)$-partite graph in which each partition has $m-1$ vertices that are independent. Thus, at most $m-1$ vertices of G are independent. It is easily verified that the clique number of G is $n-1$. It follows that $\mathcal{R}_{\mathcal{M}at\mathcal{G}}(n,m) > w$ and together with Theorem 2.2 we conclude $\mathcal{R}_{\mathcal{M}at\mathcal{G}}(n,m) = \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n,m) = w+1 = (n-1)(m-1)+1$. □

Corollary 3.17. *Let R be a commutative ring, $j \geq 2$, $n, m \geq 1$ be positive integers (n, m need not be distinct) and $S \subseteq \{X \in D \mid \det(X)$ be a proper element of $R\}$ such that $|S| \geq (n-1)(m-1)+1$.*

Then one of the following assertions hold:

(1) *There are n matrices $X_1, \ldots, X_n \in S$ such that $\det(X_1) \| \det(X_2) \| \cdots \| \det(X_n)$ (in R).*
(2) *There are m pairwise distinct matrices $Y_1, \ldots, Y_m \in S$ such that for all $1 \leq h \neq f \leq m$.*
- $\det(Y_h) \nmid \det(Y_f)$ *or*
- $\det(Y_h) \mid \det(Y_f)$ *and* $\det(Y_f) \mid \det(Y_h)$

hold.

3.5 Idempotent graphs of commutative rings

Definition 3.18. Let R be a commutative ring.
(1) We call $a \in R$ *idempotent* if $a^2 = a$.
(2) We define the *idempotent graph* $\mathrm{Idm}(R)$ of R to be the undirected simple graph with the set of idempotents of R as its vertex set and two distinct vertices a, b are adjacent if and only if $a|b$ or $b|a$.
(3) By *Idem$\mathcal{G}$* we denote the *class of all idempotent graphs*.

First, we show that the divisibility relation is a partial order on the set of idempotent elements of R.

Lemma 3.19. *Let R be a commutative ring and let V be the set of all idempotent elements of R. We define $\leq$ on V such that for all $a, b \in V$, we have $a \leq b$ if and only if $a|b$.*

Then $(V, \leq)$ is a partially ordered set and the graph $\mathrm{Idm}(R)$ is a partial order graph.

Proof. Clearly, $\leq$ is reflexive and transitive. Suppose that $a|b$ and $b|a$ (in R), that is, $a = bx$ and $b = ay$ for some $x, y \in R$. Then, since a and b are idempotent, we can conclude that

$$\begin{aligned} a - ba &= (1-b)a = (1-b)bx = bx - b^2x = bx - bx = 0 \text{ and} \\ b - ab &= (1-a)b = (1-a)ay = ay - a^2y = ay - ay = 0 \end{aligned}$$

and hence $a = ba = ab = b$, which implies that $\leq$ is anti-symmetric. □

By Lemma 3.19, it is clear that $\mathcal{R}_{Idem\mathcal{G}}(n,m) \leq \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n,m)$. Next, we show that $\mathcal{R}_{Idem\mathcal{G}}(n,m) = \mathcal{R}_{\mathcal{P}o\mathcal{G}}(n,m)$. We start with the following lemma.

Lemma 3.20. *Let R be a commutative ring and E be a set of $w \geq 3$ distinct non-trivial idempotents of R such that eR is a maximal ideal of R for every $e \in E$. Let $x = f_1 f_2 \cdots f_k$ and $y = b_1 b_2 \cdots b_j$ such that $f_1, \ldots, f_k$, $b_1, \ldots, b_j \in E$ and $2 \leq k, j < w$.*

Then

(1) *$x \neq 0$.*
(2) *$x = y$ if and only if $\{f_1, \ldots, f_k\} = \{b_1, \ldots, b_j\}$.*

Proof.
(i) Since $e_1, \ldots, e_w$ are distinct non-trivial idempotents of R and each $e_i R$ is a maximal ideal of R, $1 \leq i \leq w$, by Lemma 3.19 we conclude that $e_1 R, \ldots, e_w R$ are distinct maximal ideals of R. Since $k < w$, there exists a maximal ideal dR for some $d \in E$ such that $x = f_1 f_2 \cdots f_k \notin$ dR (note that each $f_i R$ is a maximal ideal of R). Thus, $x \neq 0$.

(ii) We may assume that $f_1 \neq b_i$ for every $1 \leq i \leq j$. Hence, $x \in f_1R$ but $y \notin f_1R$ and thus $x \neq y$. Since all f_i and b_i are idempotent elements, multiplicities have no impact, which makes the other implication obvious. □

Theorem 3.21. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct). Then for the Ramsey number $\mathcal{R}_{Idem\mathcal{G}}(n, m)$ with respect to the class of idempotent graphs the following holds:*

$$\mathcal{R}_{Idem\mathcal{G}}(n, m) = \mathcal{R}_{\mathcal{PoG}}(n, m) = (n-1)(m-1) + 1.$$

Proof. We set $w = (n-1)(m-1) \geq 1$ and show that $Idem\mathcal{G}$ contains an $(n-1)$-partite graph in which each partition consists of $m-1$ independent vertices. For this purpose, set $R = \prod_{i=1}^{w} \mathbb{Z}_2$. It is clear that R has exactly w distinct maximal ideals, say $M_1, \ldots, M_w$, and each $M_i = p_iR, 1 \leq i \leq w$ for idempotent p_i of R. We set $E = \{p_1, p_2, \ldots, p_w\}$. Note that $|E| = w$ since $p_1, p_2, \ldots, p_w$ are pairwise distinct.

For each $1 \leq i \leq n-1$, let $n_i = (i-1)(m-1)$, $a_i = p_1p_2 \cdots p_{n_i}$ (hence $a_1 = 1$) and $A_i = \{a_ip_{n_i+1}, \ldots, a_ip_{n_i+(m-1)}\}$. Note that $A_1 = \{p_1, \ldots, p_{m-1}\}$.

By construction of each A_i and in light of Lemma 3.20, for each $1 \leq i \leq n-1$, we have $|A_i| = m-1$ and the vertices of A_i are independent. Let H be the subgraph of $Idm(R)$, which is induced by $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$.

By construction of H and Lemma 3.20, we conclude that H is a complete $(n-1)$-partite graph in which each partition has $m-1$ vertices that are independent. Thus, H has exactly $m-1$ vertices that are independent. It is easily verified that the clique number of H is $n-1$. Thus, $\mathcal{R}_{Idem\mathcal{G}}(n, m) > w$. Hence by Theorem 2.2, we have $\mathcal{R}_{Idem\mathcal{G}}(n, m) = \mathcal{R}_{\mathcal{PoG}}(n, m) = w + 1 = (n-1)(m-1) + 1$. □

Remark 3.22. Observe that the ring $R = \prod_{i=1}^{w} \mathbb{Z}_2$ in the proof of Theorem 3.21 is a finite boolean ring. Let $Bool\mathcal{G}$ denote the subclass of $Idem\mathcal{G}$ consisting of all idempotent graphs of boolean rings.

In view of the proof of Theorem 3.21, we conclude that $\mathcal{R}_{Bool\mathcal{G}}(n, m) = \mathcal{R}_{Idem\mathcal{G}}(n, m)$. Thus, we state this result without a proof.

Theorem 3.23. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct).*
Then $\mathcal{R}_{Bool\mathcal{G}}(n, m) = \mathcal{R}_{Idem\mathcal{G}}(n, m) = \mathcal{R}_{\mathcal{PoG}}(n, m) = (n-1)(m-1) + 1$.

In view of Theorem 3.21, we have the following result.

Corollary 3.24. *Let $n, m \geq 1$ be positive integers (n, m need not be distinct), $k = (n-1)(m-1) + 1$ and A be a subset of idempotent elements of R such that $|A| \geq k$.*
Then one of the following assertions hold:
(1) *There are n pairwise distinct elements (distinct idempotents) $a_1, \ldots, a_n \in A$ such that $a_1|a_2|\cdots|a_n$ (in R).*
(2) *There are m pairwise distinct elements (distinct idempotents) $b_1, \ldots, b_m \in A$ such that $b_h \nmid b_f$ (in R) for all $1 \leq h \neq f \leq m$.*

4 An example class C of partial order graphs with $\mathcal{R}_C(n, m) \neq \mathcal{R}_C(m, n)$

In this section, we present a subclass C of $\mathcal{PDG}$ with respect to which the Ramsey numbers $\mathcal{R}_C$ are non-symmetric in m and n. We recall the following definition [11].

Definition 4.1. A subset S of a ring R is called a *positive semi-cone* of R if S satisfies the following conditions:
(1) $S \cap (-S) = \{0\}$.
(2) $S + S \subseteq S$.
(3) $S \cdot S \subseteq S$.

If S satisfies the aforementioned conditions and $S \cup (-S) = R$, then S is called a *positive cone* of R [12].

For a positive semi-cone S of R, define $\leq_S$ on R such that for all $a, b \in R$, we have $a \leq_S b$ if and only if $b - a \in S$. Then $(R, \leq_S)$ is a partially ordered set. We define the *S-positive semi-cone graph* $\mathrm{ConeG}_S(R)$ of R to be the simple, undirected graph with vertex set R such that two vertices a, b are connected by an edge if and only if $b - a \in S$ or $a - b \in S$. Then Cone $G_S(R)$ is a partial ordered graph.

Definition 4.2. For $k \geq 2$, let $P_k = \{0, k, 2k, 3k, \ldots\} = k\mathbb{N}_0$.
(1) For $a, b \in \mathbb{Z}$ we define $a \leq_k b$ if and only if $b - a \in P_k$.
(2) We define the *k-positive semi-cone graph* $\mathrm{ConeG}_k(\mathbb{Z})$ of $\mathbb{Z}$ to be the simple, undirected graph with vertex set $\mathbb{Z}$ such that two vertices $a, b \in R$ are connected by an edge if and only if $|a - b| \in P_k$.
(3) For every positive integer $k \geq 2$, let $\mathcal{R}_{k\text{-}ConeG}(n, m)$ be the minimal number of vertices r such that every induced subgraph of the partial order graph $\mathrm{ConeG}_k(\mathbb{Z})$ consisting of r vertices contains either a complete n-clique K_n or an independent set consisting of m vertices.

Remark 4.3.
(1) For every $k \geq 2$, P_k is a positive semi-cone subset of $\mathbb{Z}$ that is not a positive cone of $\mathbb{Z}$. The relation $a \leq_k b$ if and only if $b - a \in P_k$ is a partial order on $\mathbb{Z}$ and $\mathrm{ConeG}_k(\mathbb{Z})$ is a partial order graph.
(2) For every $k \geq 2$, two vertices a, b of $\mathrm{ConeG}_k(\mathbb{Z})$ are connected by an edge if and only if $a \equiv b \pmod{k}$.

For each $k \geq 2$, the following theorem shows that $\mathcal{R}_{k\text{-}ConeG}(n, m)$ is not always symmetric in m and n.

Theorem 4.4. *Let $k \geq 2$, $n, m \geq 1$ be positive integers (n, m need not be distinct). Then*
(1) *If $1 \leq m \leq k + 1$, then*

$$\mathcal{R}_{k\text{-}ConeG}(n, m) = (n - 1)(m - 1) + 1.$$

In particular, if $1 \leq n, m \leq k + 1$, then $\mathcal{R}_{k\text{-}ConeG}(n, m) = \mathcal{R}_{k\text{-}ConeG}(m, n) = (n - 1)(m - 1) + 1$ is symmetric in n and m.
(2) *If $m > k + 1$, then*

$$\mathcal{R}_{k\text{-}ConeG}(n, m) = \mathcal{R}_{k\text{-}ConeG}(n, k + 1) = (n - 1)k + 1$$

only depends on the first argument n. In particular, assume that $n \neq m$. If $n > k + 1$ or $m > k + 1$, then $\mathcal{R}_{k\text{-}ConeG}(n, m) \neq \mathcal{R}_{k\text{-}ConeG}(m, n)$.

Proof. (1) For $n = 1$ or $m = 1$, the assertion immediately follows, so we assume $n \geq 2$ and $2 \leq m \leq k + 1$. For each $1 \leq i \leq m - 1$, let

$$A_i = \{k + i, 2k + i, \ldots, (n - 1)k + i\}.$$

By construction, each A_i contains $n - 1$ distinct elements a with $a - i \in P_k$. Therefore, for $a \neq b \in A_i$, either $b - a \in P_k$ or $a - b \in P_k$ and hence each A_i induces a complete subgraph of $\mathrm{ConeG}_k(\mathbb{Z})$ with exactly $n - 1$ vertices. Moreover, since $m - 1 \leq k$, for $a \in A_i$ and $b \in A_j$ with $1 \leq i \neq j \leq m - 1$, then $a \not\equiv b \pmod{k}$ and therefore a and b are not connected by an edge.

Let H be the subgraph of $\mathrm{ConeG}_k(\mathbb{Z})$ which is induced by the vertex set $A_1 \cup \cdots \cup A_{m-1}$. Then H is disjoint union of $m - 1$ $(n - 1)$-cliques and hence does neither contain an n-clique nor an independent set of cardinality m, which implies that $\mathcal{R}_{k\text{-}ConeG}(n, m) > (n - 1)(m - 1)$. It now follows from Theorem 2.2 that $\mathcal{R}_{k\text{-}ConeG}(n, m) = (n - 1)(m - 1) + 1$.

The symmetry assertion follows immediately from this if, moreover, $1 \leq n \leq k + 1$ holds.

(2) Recall that two vertices a, b of $\mathrm{ConeG}_k(\mathbb{Z})$ are connected by an edge if and only if $a \equiv b \pmod{k}$. Therefore, a maximal independent subset has cardinality k (the number of residue class modulo k). Thus if $m \geq k + 1$, then $\mathrm{ConeG}_k(\mathbb{Z})$ cannot contain an independent set with m distinct vertices. Therefore, for all $m \geq k + 1$, the equality

$$\mathcal{R}_{k\text{-}Cone\mathcal{G}}(n, m) = \mathcal{R}_{k\text{-}Cone\mathcal{G}}(n, k + 1)$$

holds and the assertion now follows from (1). □

In view of Theorem 4.4, we have the following result.

Corollary 4.5. *Let $k \geq 2$ and $n, m \geq 1$ be positive integers (n, m need not be distinct) and A be a subset of $\mathbb{Z}$. Then*

(1) *If $2 \leq m \leq k + 1$ and $|A| > (n - 1)(m - 1)$, then there are at least n pairwise distinct elements $a_1, \ldots, a_n \in A$ such that $a_1 \equiv \cdots \equiv a_n \pmod{k}$ or there at least m elements $b_1, \ldots, b_m \in A$ such that $b_i \not\equiv b_j \pmod{k}$ for all $1 \leq i \neq j \leq m$.*

(2) *If $m > k + 1$ and $|A| > (n - 1)k$, then there are at least n pairwise distinct elements of A, say $a_1, \ldots, a_n$ such that $a_1 \equiv \cdots \equiv a_n \pmod{k}$.*

Example 4.6. The induced subgraph H of $\mathrm{ConeG}_3(\mathbb{Z})$ with vertex set $V = \{1, 2, 3, \ldots, 12\}$ consists of three 4-cliques. Since $|V| = 12$, H satisfies $\mathcal{R}_{3\text{-Cone}\mathcal{G}}(12, 2)$, $\mathcal{R}_{3\text{-Cone}\mathcal{G}}(4, 4)$, $\mathcal{R}_{3\text{-Cone}\mathcal{G}}(6, 3)$, $\mathcal{R}_{3\text{-Cone}\mathcal{G}}(5, 3)$ and $\mathcal{R}_{3\text{-Cone}\mathcal{G}}(4, 10)$, see Figure 5.

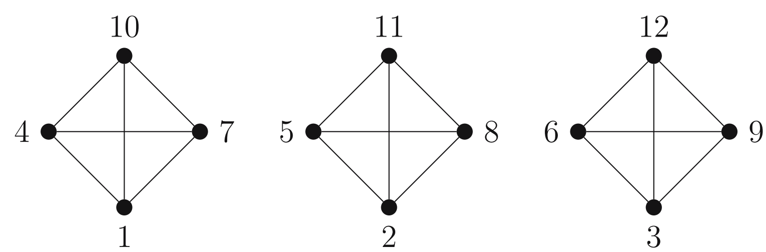

Figure 5: Subgraph H of $\mathrm{ConeG}_3(\mathbb{Z})$ induced by the vertices $V = \{1, 2, 3, \ldots, 12\}$.

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